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Rational Bubbles and the S&P 500. An empirical approach

Burbujas racionales y el S&P 500. Una metodología empírica

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Abstract

We analyze if the dynamics of the S&P500 resemble those of a rational bubble. We find positive evidence in this question by applying the Kalman Filter to a suitable asset pricing model proposed and our conclusion is robust to three different stochastic discount factors SDFs considered: Linear Utility, Log Utility and CRRA utility. We also find evidence of a relationship between the type of SDF and the size of a bubble in the S&P500 case.

Keywords: Bubble Estimation; Kalman Filter; Stochastic Discount Factor.

Resumen

El presente documento analiza si la evolución del S&P500 se parece a la de una burbuja racional. Encontramos evidencia positiva en esta interrogante a través de la aplicación del Filtro de Kalman a un modelo de valoración de acciones propuesto, y nuestra conclusión es robusta empleando tres diferentes factores estocásticos de descuento: utilidad lineal, utilidad logarítmica y utilidad CRRA. Encontramos también evidencia de la existencia de una relación entre el tipo de factor estocástico de descuento y el tamaño de la burbuja.

Palabras clave: Estimación de burbujas; Filtro de Kalman; Factor estocástico de descuento.

Classification/Clasificación JEL: G12, C13, C32

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1. Introduction

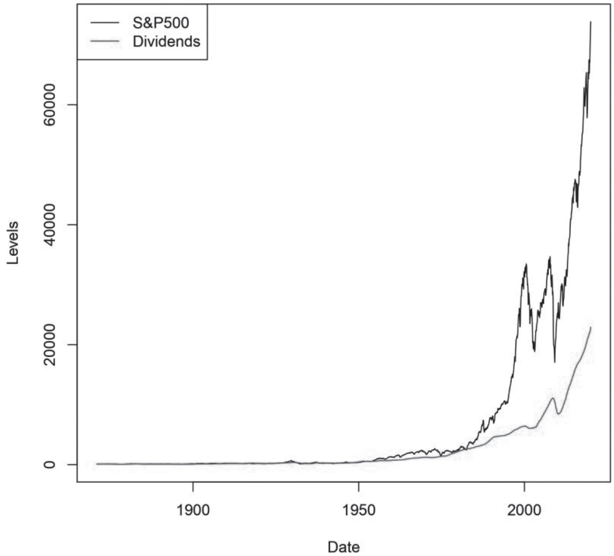
The US Stock Market is one of the principal financial markets in the world. It's well represented by the S&P500 Index, which gives the behaviour of US stock prices in general being a leading indicator of the financial and economic situation in the US. For Latinamerican countries, the financial health of the US is important to follow since it has immediate contagion effect in important financial markets like Mexico, Brazil and Chile. As a matter of fact, Uribe and Mejía (2014) show that a bubble in US anticipates bubbles in the main emerging markets in Latinamerica as a contagion effect. The reason is that when a bubble bursts in developed financial markets like US, capital flows to emerging markets economies or commodities like petroleum or gold. Because of this, it is relevant for Latinamerica countries to study the bubble component in stock markets in US.

As shown in a simple representative agent model in Gürkaynak (2008) the stock price pt can be represented as:

$$p_t = f_t + b_t \tag{1}$$

Being f_t the fundamental component and b_t the bubble component. The latter can be thought as a pyramid scheme, it is reasonable to invest in it as long as it is expected that other people will also invest in it. In the case of stocks, f_t is mainly driven by dividends d_t and we see in Figure 1 that there has been a decoupling from prices and dividends in the last 20 years. Diba and Grossman (1988) conclude that for the non-existence of a bubble in stock prices both series should co-move or be cointegrated if they have a unit root. With a sample that goes until early 90s, they reject the presence of a bubble in stock prices but it is clear from Figure 1 that the important divergence starts just before 2000. Therefore, we might have some scepticism against the conclusion of Diba and Grossman (1988).

Figure 1: Evolution of S&P500 Levels and Dividends. Year 1871 = 100



Source: Own elaboration.

We can go one step further and ask if b_t has become almost the main driving force behind stock the S&P500 behaviour. We can start by assuming that the S&P500 follows the dynamics of a rational bubble.

$$p_t = \frac{E_t\left[p_{t+1}\right]}{\left(1+r_t\right)} \tag{2}$$

This is a non-arbitrage condition for a rational bubble and r_t is the risk-free interest rate. We can rewrite equation 2 as:

$$p_{t+1} = \left(1+r_t\right) p_t + u_{t+1} \tag{3}$$

In equation 3, if it is true that a rational bubble explains the dynamics of S&P500 then u_{t+1}, should be stationary. If p_{t+1} and $(1+r_t)\,p_t$ have a unit root then they should be cointegrated in order to verify this rational bubble equation. We check this claim empirically[1].

In Table 1 we verify that both p_{t+1} and $(1+r_t)\,p_t$ have a unit root[2]. Additionally, we check that u_{t+1} is stationary, which confirms the hypothesis that p_{t+1} and $(1+r_t)\,p_t$ are cointegrated and the rational bubble equation is verified.

Table 1
Sims Unit Root Test for p_{t+1}, $(1+r_t)\,p_t$ and u_{t+1}

Variable	τ^2	Critical	Unit Root
p_{t+1}	0.13	13.46	Yes
$(1+r_t)\,p_t$	0.13	13.46	Yes
u_{t+1}	511	6.63	No

Source: Own elaboration.

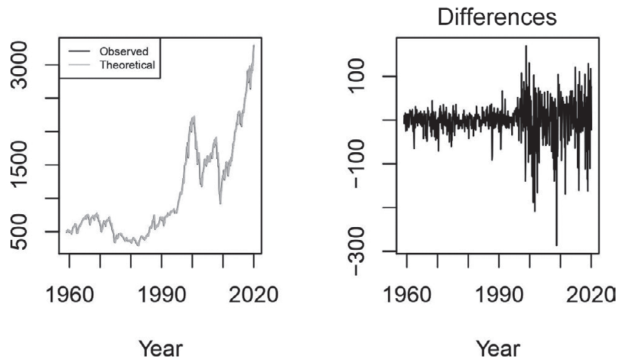
In Figure 2, we plot the observed levels of the S&P500 with the theoretical levels from the rational bubble postulated in equation 3. We can see both series co-move.

One possible caveat of the previous analysis is that by using monthly data, information is rapidly incorporated in prices the shorter the period of analysis is. Thus, we repeat the analysis by using yearly data to avoid the fast incorporation of information into prices. Figure 3 shows the results:

1 We use monthly data from January 1959 to April 2020 for S&P500 levels that is in Robert Shiller web page. Additionally, we obtain the risk-free rate (1-month T-Bill Rate) from Kenneth French web page. We express the variables in real terms.

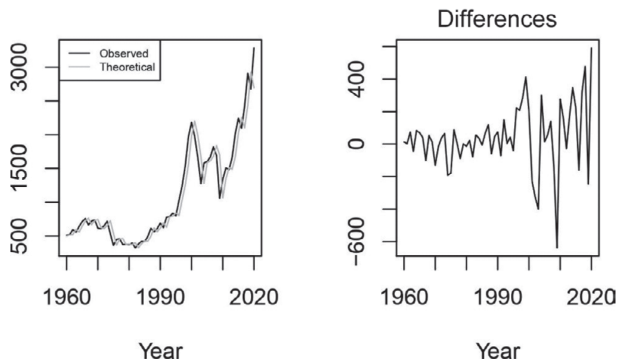
2 We use the Bayesian Unit Root Test proposed by Sims (1988). We use a prior probability that the process has a unit root of $\alpha = 0.5. A$ dditionally, the critical value was computed for large samples.

Figure 2: Observed and Theoretical Monthly Levels of S&P500



Source: Own elaboration.

Figure 3: Observed and Theoretical Yearly Levels of S&P500



Source: Own elaboration.

The difference between the observed levels of the S&P500 and the theoretical ones can be clearly spotted. However, it is still true that a great portion of the dynamics of S&P500 is explained by the dynamics of a rational bubble. This again rises the doubt of imposing the restriction of $b_t = 0$ in asset pricing models for stocks, in this particular case for the S&P500.

Most of the empirical literature of bubbles has focused in developing econometric tests for assessing the validity of $b_t \neq 0$. This is summarized in Gürkaynak (2008). The contributions can be divided in four streams:

1. Variance Bound Test. In this literature, a linear utility function is assumed so the usual present value model for stock prices holds. The conclusion of this literature is that the theoretical variance of stock prices under the present value model is bigger than the variance of observed stock prices. If this is not the case, an argument in favour of the existence of the bubble component rises. The main critic is that there might not be a bubble in stock prices but the present value model could not be the correct one.
2. Specification Test of West. This is the first specific test for a bubble since it has in the alternative hypothesis that a bubble is present in stock prices. It assumes linear utility as well to model the fundamental component of the stock price. Two elements are used for this: the main asset pricing equation through the known Euler equation and an autoregressive process for the dividends. The second element is a regression of stock prices on dividends. If the set of parameters is equal statistically in these two approaches; then there is evidence of no bubble; whereas the opposite points out to the existence of a rational bubble in the data. The main contribution of this test is to isolate the modeling of the fundamental from the existence of a bubble.
3. Unit Root and Cointegration Tests. Diba and Grossman (1988) pointed out that if stock prices and dividends are unit root processes then they should co-move or be cointegrated in order to discard the presence of a bubble. With a sample until late 90s, they reject the existence of a bubble in stock prices. However, Evans (1991) pointed out that the use of traditional unit root and cointegration tests can reject the presence of periodically explosive bubbles in the data. To overcome this pitfall, Norden (1996) uses regime switching models to account for periodically explosive bubbles. More recently, Phillips *et al.* (2011) developed a test for mildly explosive processes and applied it to the detection of bubbles. They show that their procedure is robust to the Evans (1991) critique. Finally, in Latinamerica Uribe and Mejía (2014) use the sign test for random walks developed by So and Shin (2001) as a bubble test and they argue that it is robust to the Evans (1991) critique.
4. Bubbles as Non-observables. These are not bubble tests since they assume that a bubble is present in stock prices but they develop procedures for estimating the bubble component. The pioneer is Wu (1997) who applies the Kalman Filter to the linear utility model of stock prices and retrieves the bubble component. This paper is in this section of the literature since we also use the Kalman Filter to extract the bubble component, the main

difference with Wu (1997) is that we use a more general asset pricing model for stocks that includes a borrowing constraint and we consider three different utility functions: linear, logarithmic and CRRA.

The main issue with this early empirical literature is that we are dealing with two unobserved components: f_t and b_t. Thus the acceptance or rejection of $b_t \neq 0$ by this early tests can be a byproduct of a incomplete model for f_t. In other words, some authors refuted the existence of bubbles in the stock market by specifying a more complex model for f_t; nevertheless, given the behaviour of dividends in the last 20 years, it is hard to adjust a complex model that closes the gap between stock prices and dividends.

This paper is not intended as a proposal for an alternative econometric method to asses whether $b_t \neq 0$ or not. Given the empirical evidence showed in this introduction we consider that b_t cannot be neglected. The issue is how important its role is. For this, we apply the method used in Wu (1997) through the Kalman Filter to a simple asset price model with borrowing constraints and with the characteristic of considering different utility functions that derive in different stochastic discount factors. We find robust evidence that the b_t component has similar dynamics to the observed S&P500 independent of the stochastic discount factor considered. Thus, a rational bubble model for the S&P500 might seem a suitable one.

Section 2 develops the theoretical model. Section 3 shows the empirical results and Section 4 concludes.

2. Model

Time is discrete and runs from $t = 0, \ldots, \infty$. There are two cohorts of investors in an economy: young and old. Following Martin and Ventura (2016), investors only focus on consumption when they are old. When young, they receive an endowment e_t and borrow from financial markets an amount b_t. Borrowing is restricted in this economy and a borrowing constraint is imposed. Young investors can use these funds received in youth to buy a stock at price p_t. This stock gives them, when old, funds from the gain or loss of value for prices p_{t+1} a$_n$d a dividend d_t already known a time t. The funds received when old are used to repay the debt and to consume c_{t+1}.

The borrowing constraint plays a crucial role in this model. The parameter ϕ measures the degree of development of the financial markets. The lower the ϕ, the more difficult to borrow in this economy and the more important role of stock prices to help increase borrowing. This element of the model allows that in equilibrium stock prices not only have a fundamental value from the dividend stream they have but also an additional value for the relaxing of the borrowing constraint. This ensures that a bubble is sustained in equilibrium. It is not the purpose of the paper to show this but we need this enviroment in order to have a bubble in stock prices. Finally, the utility function fulfills the usual Inada conditions. The model is:

$$\begin{array}{cc} \max\limits_{c_t, s_{t+1}, b_t} & \beta E_t\left[u\left(c_{t+1}\right)\right] \\ p_t s_{t+1} & = b_t + e_t \\ c_{t+1} + \left(1 + r_t\right) b_t & = \left(p_{t+1} + d_t\right) s_{t+1} \\ -b_t & \leq \phi p_t s_{t+1} \end{array} \tag{4}$$

The FOCs for this problem are[3]:

$$p_t = E_t\left[\frac{\beta u'\left(c_{t+1}\right)}{\left(u'\left(c_t\right) - \phi \mu_t\right)}\left(p_{t+1} + d_t\right)\right] \tag{5}$$

$$\mu_t = \beta E_t\left[u'\left(c_{t+1}\right)\right]\left(1 + r_t\right) - u'\left(c_t\right) \tag{6}$$

where μ_t is the lagrange multiplier from the borrowing constraint. We can define the stochastic discount factor SDF as:

$$m_{t+1} = \frac{\beta u'\left(c_{t+1}\right)}{\left(u'\left(c_t\right) - \phi \mu_t\right)} \tag{7}$$

3 The borrowing constraint binds in equilibrium.

Thus the pricing equation would be:

$$p_t = E_t\left[m_{t+1}\left(p_{t+1}+d_t\right)\right] \tag{8}$$

To ease the analysis we apply the Taylor approximation of order one to the term $m_{t+1}\left(p_{t+1}+d_t\right)$[4] around the median of p_{t+1} which we call $\overline{p}$, the median of m_{t+1} which we call $\overline{m}$ and finally the median of d_t called $\overline{d}$. The expression in equation 8 can be rewritten as:

$$p_t = \overline{m}d_t - \overline{m}\left(\overline{p}+\overline{d}\right) + \left(\overline{p}+\overline{d}\right)E_t\left[m_{t+1}\right] + \overline{m}E_t\left[p_{t+1}\right] \tag{9}$$

Forward updating equation 9 we obtain:

$$p_t = \underbrace{\sum_{j=0}^{\infty}\overline{m}^{j+1}E_t\left[d_{t+j}\right] - \left(\overline{p}+\overline{d}\right)\sum_{j=1}^{\infty}\overline{m}^{j} + \left(\overline{p}+\overline{d}\right)\sum_{j=0}^{\infty}\overline{m}^{j}E_t\left[m_{t+j+1}\right]}_{f_t} + \underbrace{\lim_{j\to\infty}\overline{m}^{j}E_t\left[p_{t+j}\right]}_{b_t} \tag{10}$$

The last term of equation 10 is the bubble term b_t[5] which follows the dynamics:

$$E_t\left[b_{t+1}\right] = \frac{1}{\overline{m}}b_t \tag{11}$$

Equation 11 represents the non-arbitrage condition for the rational bubble of the model we specified at the beginning of the section. This will be estimated by the Kalman Filter but there is an important restriction that $b_t >= 0$ because of free-disposal. To avoid complications in the application of the Kalman Filter to equation 10 since we will have to impose the restriction of non-negativity of the bubble term, we take the model in first differences as an input. In this

4 It is not necessary to go beyond the Taylor approximation of order 1 since higher derivatives $m_{t+1}\left(p_{t+1}+d_t\right)$ are 0.

5 We also called b_t the borrowing the young investor gets from financial markets. This is an abuse of notation but it is worth clarifying it.

case, we do not need to impose any restriction. The final model to be used in the estimation for the bubble for the S&P500 will be:

$$\Delta p_t = \sum_{j=0}^{\infty} \overline{m}^{j+1}\left(E_t\left[d_{t+j}\right] - E_{t-1}\left[d_{t+j-1}\right]\right) + \left(\overline{p} + \overline{d}\right)\sum_{j=0}^{\infty} \overline{m}^{j}\left(E_t\left[m_{t+j+1}\right] - E_{t-1}\left[m_{t+j}\right]\right) + \Delta b_t \quad (12)$$

3. Results

Before starting the analysis, it is worth mentioning the source of the data. Stock prices (levels of the S&P500), dividends and Consumer Price Index (CPI) came from the personal web page of Robert Shiller. The risk-free rate came from the Kenneth French web page where he published about the Fama-French factors. The risk-free rate is the interest rate in the 1-month T-Bill. Finally, the consumption data is the Personal Consumption Expenditures in billions of dollars from the web page of the Federal Reserve Bank of St. Louis. All data is monthly from January 1960 to February 2020. The data was expressed in real terms using the base CPI as of February 2020.

3.1. Linear Utility Model

For the Linear Utility Model (LUM) we specify the following utility function:

$$u\left(c_t\right) = c_t \quad (13)$$

We start the analysis with the stock price equation derived in equation 12. Unfortunately, this is a complicated equation to put it directly into the Kalman Filter. We need to simplify it in some way. To do this, we need to find first the ARIMA order[6] of the dividend process d_t and for the SDF component m_t. In case of this last variable, we use reasonable values of the degree of impatience $\beta = 0.95$ and the development of financial markets $\phi = 0.5$. Table 2 shows the results.

6 We only consider here the autoregressive component and the integrated one as done in Wu (1997). To find out about the order of integration we apply the Bayesian Unit Root Test of Sims (1988)

Table 2
ARIMA models for d_t and m_t

Variable	Model
d_t	ARIMA(4, 1, 0)
m_t	ARIMA(3, 0, 0)

Source: Own elaboration.

Taking into account these results, the main asset pricing equation of the LU model is[7]:

$$\begin{aligned}\Delta p_t = {} & \frac{\bar{m}}{1-\bar{m}}\Delta d_t + \frac{\sum_{j=1}^{4}\bar{m}^{j+1}\alpha_j}{(1-\bar{m})\left(1-\sum_{j=1}^{4}\bar{m}^j\alpha_j\right)}\Delta^2 d_t + \frac{\sum_{j=2}^{4}\bar{m}^{j}\alpha_j}{(1-\bar{m})\left(1-\sum_{j=1}^{4}\bar{m}^j\alpha_j\right)}\Delta^2 d_{t-1} + \\ & \frac{\sum_{j=3}^{4}\bar{m}^{j-1}\alpha_j}{(1-\bar{m})\left(1-\sum_{j=1}^{4}\bar{m}^j\alpha_j\right)}\Delta^2 d_{t-2} + \frac{\bar{m}^2\alpha_4}{(1-\bar{m})\left(1-\sum_{j=1}^{4}\bar{m}^j\alpha_j\right)}\Delta^2 d_{t-3} + \\ & \frac{\sum_{j=1}^{3}\bar{m}^{j-1}\psi_j}{1-\sum_{j=1}^{3}\bar{m}^j\psi_j}\left(\bar{p}+\bar{m}\right)\Delta m_t + \frac{\psi_2+\bar{m}\psi_3}{1-\sum_{j=1}^{3}\bar{m}^j\psi_j}\left(\bar{p}+\bar{m}\right)\Delta m_{t-1} + \\ & \frac{\psi_3}{1-\sum_{j=1}^{3}\bar{m}^j\psi_j}\left(\bar{p}+\bar{m}\right)\Delta m_{t-2} + \Delta b_t\end{aligned} \tag{14}$$

Along with equation 14, we should consider into the measurement system both equations of the dynamics of d_t and m_t which are:

$$\Delta^2 d_{t+1} = \alpha_1\Delta^2 d_t + \alpha_2\Delta^2 d_{t-1} + \alpha_3\Delta^2 d_{t-2} + \alpha_4\Delta^2 d_{t-3} + e_{(1,t+1)} \tag{15}$$

$$\Delta m_{t+1} = \psi_1\Delta m_t + \psi_2\Delta m_{t-1} + \psi_3\Delta m_{t-2} + e_{(2,t+1)} \tag{16}$$

Finally, in order to apply the Kalman Filter we need the state equation that corresponds to the bubble dynamics:

7 There is a detailed derivation of this simplified form in Appendix A.

$$\Delta b_{t+1} = \frac{1}{\bar{m}} \Delta b_t + w_{t+1} \tag{17}$$

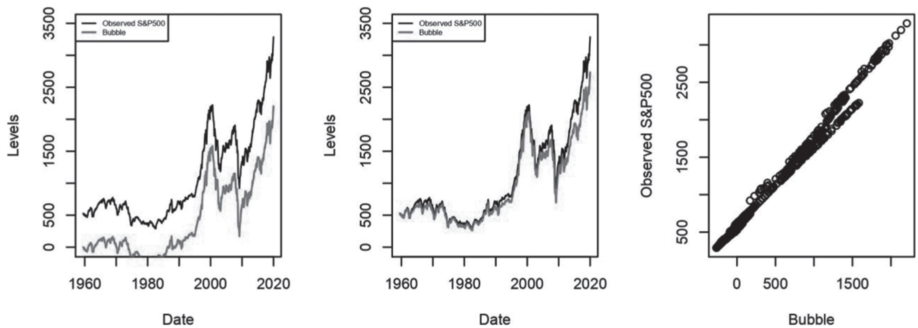
Before starting the Kalman Filter, we should consider the initial forecasted value for the estimated bubble $\Delta\tilde{b}_{1|0}$. We attach to the idea exposed in Santos and Woodford (1997) and Diba and Grossman (1988) that the bubble started at the same time of the stock. Thus the bubble is always present. We consider that the beginning of the bubble is the start of our sample which is January 1960. Therefore, the initial value for the Kalman Filter should be $\Delta\tilde{b}_{1|0} = 0$[8].

We wait for a later subsection of the results to show the coefficients estimated. In Figure 4 we plot the results of the estimated bubble considering a linear utility model and the levels of the S&P500. Remember that in the Kalman Filter applied we estimated $\Delta\tilde{b}_t$ so to construct $\tilde{b}_t$ we need an initial value for the bubble. We can peek any positive value (the bubble is non-negative because of free-disposal) and construct the bubble with the estimated $\Delta\tilde{b}_t$. This leaves us with multiple bubbles to be considered and that is a characteristic of models with bubbles: they have multiple equilibria. Finally, the presented results are the smoothed values of the estimated bubble.

To begin with, at the left plot of Figure 4, we consider an initial value of $\tilde{b}_1 = 1$ and we clearly see that the dynamics of the estimated bubble are very similar to the dynamics of the observed levels of the S&P500. We can look this more clearly by choosing an initial value of the bubble $\tilde{b}_1$ equal to the level of the S&P500 in January 1960, and we see in the middle plot of Figure 4, that both series are very much the same. Finally, in the right plot of Figure 4, we do a scatterplot of the levels of S&P500 and the estimated bubble with an initial value of $\tilde{b}_1 = 1$. There is a direct and strong relationship between both.

8 In the case of the MSE for the estimated bubble we consider a lot of uncertainty with an initial value of $\tilde{P}_{1|0} = 100$.

Figure 4: Observed S&P500 Levels and Estimated Linear Utility Bubble



Source: Own elaboration.

Now we repeat this procedure with the log utility and the CRRA utility case and see if the results are similar.

3.2. Logarithmic Utility

We consider an utility function in the form:

$$u(c_t) = \log(c_t) \tag{18}$$

To begin the analysis we need to determine the ARIMA order of $g_t = c_t^{-1}$ in order to estimate $E_t[g_{t+1}]$ that belongs to the Lagrange multiplier μ_t in equation 6. Table 3 shows the result.

Table 3
ARIMA models for g_t

Variable	Model
g_t	ARIMA(8, 1, 0)

Source: Own elaboration.

With this result, $E_t[g_{t+1}]$ can be computed as follows:

$$E_t[g_{t+1}] = g_t + \sum_{j=1}^{8} \rho_j \Delta g_{t+1-j} \tag{19}$$

Now we can estimate the SDF m_t (taking into account reasonable values of $\beta = 0.95$ and the degree of financial openness $\phi = 0.5$) and find its ARIMA order. In the case of the dividends d_t, we use the results showed in Table 2. Table 4 shows the results for m_t.

Table 4
ARIMA models for m_t.

Variable	Model
m_t	ARIMA(8, 0, 0)

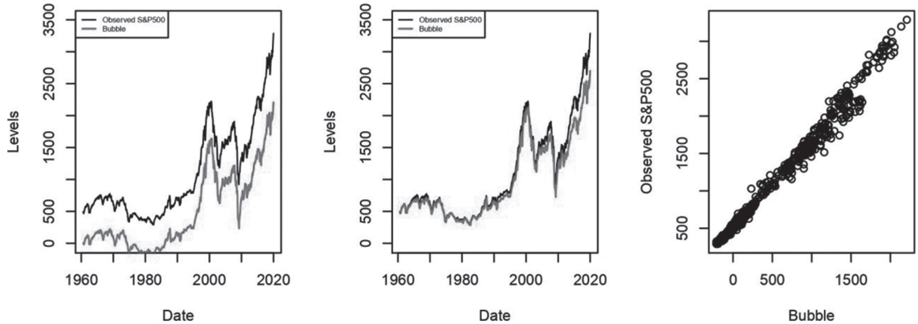
Source: Own elaboration.

With this information we are in the position to derive the main asset pricing equation of the Logarithmic Utility Model along with the dynamics of dividends and the SDF. We do not show the equations because they are similar in essence to the ones derived for the Linear Utility Model (equations 14 to 15) and we additionally use the bubble dynamics equation (Equation 17).

As discussed in the Linear Utility results, we consider an initial value for the forecasted $\Delta\tilde{b}_{1|0} = 0$ [9]. The results presented are the smoothed values of the estimated bubble. In Figure 5, we present the estimated bubble under the Log Utility function and compare it to the observed levels of the S&P500. As mentioned in the Linear Utility case, we need to define an initial value for the bubble since we estimated $\Delta\tilde{b}_{t|t}$. This initial value has to be positive since bubbles are positive because of free disposal. On the graph of the left, with an initial value of $\tilde{b}_{1|1} = 1$, we compare the dynamics of the estimated bubble with the observed levels of the S&P500. We can notice that the dynamics of both series are alike. To highlight this, in the middle graph of Figure 5, we define an initial bubble level $\tilde{b}_{1|1}$ equal to the observed level of the S&P500 in January 1960. We see that the estimated bubble follows closely the observed levels of the S&P500. Finally, the graph on the right confirms the direct and strong relationship between the estimated bubble and the observed levels of the S&P500.

9 A lot of uncertainty was also considered initially on the estimates from the Kalman Filter thus the value for the initial forecasted state was $P_{1|0} = 100$.

Figure 5: Observed S&P500 Levels and Estimated Log Utility Bubble



Source: Own elaboration.

So far, the results are similar for the Linear Utility bubble and the Log Utility bubble. We now consider the case of a CRRA Utility function.

3.3. CRRA Utility

In this section, we consider an utility function in the form:

$$u\left(c_t\right)=\frac{c_t^{1-\gamma}-1}{1-\gamma}\gamma>1 \tag{20}$$

We define $g_t = c_t^{-\gamma}$ and adopt a reasonable relative risk aversion coefficient $\gamma = 3$. As we did with the Logarithmic Utility case, we need to compute $E_t\left[g_{t+1}\right]$ in order to estimate the Lagrange multiplier μ_t of equation 6 and ultimately the SDF m_t. We present the ARIMA order of g_t and m_t in Table 5.

Table 5
ARIMA models for g_t and m_t.

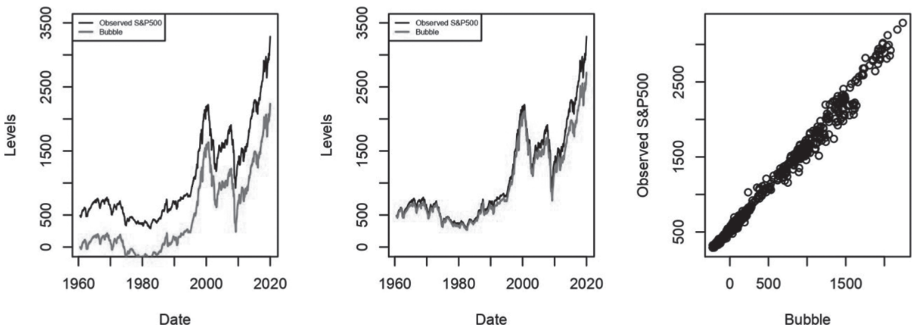
Variable	Model
g_t	ARIMA(7, 1, 0)
g_t	ARIMA(8, 0, 0)

Source: Own elaboration.

With this information, we can derive the main asset pricing equation and the dynamics of d_t and m_t. Because these equations are similar in essence to equations 14 to 16 but longer in extension we omit them here. Bubble dynamics are considered for the state equation in the Kalman Filter as is showed in equation 17.

As done in previous analysis, we initialize the forecasted state $\Delta\tilde{b}_{1|0} = 0$ and give a big initial MSE to it $P_{1|0} = 100$. In Figure 6, we display the results. We mentioned that we can have multiple estimated bubbles $\tilde{b}_t$ depending on the initial value but the dynamics are similar and they resemble the dynamics of the Observed S&P500. We can see this point in the graph of the left and middle of Figure 6. In the first case, we chose an initial value $\tilde{b}_1 = 1$ and in the second we chose an initial value equal to the observed level of the S&P500 for January 1960. We clearly see that the series dynamics resemble each other, and this result is confirmed by the scatterplot of the right which shows a direct relationship between the observed levels of the S&P500 and the estimated bubble.

Figure 6: Observed S&P500 Levels and Estimated CRRA Utility Bubble



Source: Own elaboration.

So far, the results of the three models are similar but it is useful to contrast them. We do this in the next subsection.

3.4. Results comparison

To begin with, we compare the estimated coefficients across the three models. Table 6 shows the results. We see that the estimations are kind of robust across the three models and almost all the coefficients are significative at 5% level. Finally, as we move across the models from linear to CRRA, the estimated volatility of the bubble increases and in the three models is significative.

Table 6
Estimated coefficients and standard errors

Variable	Coeff.LU	SE.LU	Coeff.Log	SE.Log	Coeff. CRRA	SE.CRRA
α_1	-0.19326	0.03721	-0.18881	0.03789	-0.18555	0.03798
α_2	-0.18264	0.03651	-0.20005	0.03712	-0.19921	0.03736
α_3	-0.23039	0.03626	-0.22606	0.03707	-0.22269	0.03708
α_4	-0.07171	0.03668	-0.08713	0.03689	-0.08583	0.03722
ψ_1	-0.00027	0.00005	-0.00147	0.00005	-0.00147	0.00005
ψ_2	-0.00019	0.00005	-0.00131	0.00007	-0.00137	0.00007
ψ_3	0.00003	0.00005	-0.00114	0.00008	-0.00121	0.00009
ψ_4	NA	NA	-0.00102	0.00009	-0.00109	0.00009
ψ_5	NA	NA	-0.00088	0.00009	-0.00096	0.00009
ψ_6	NA	NA	-0.00064	0.00008	-0.00066	0.00008
ψ_7	NA	NA	-0.00033	0.00007	-0.00025	0.00007
ψ_8	NA	NA	-0.00009	0.00005	-0.00006	0.00005
σ_d	0.01117	0.00059	0.01111	0.00059	0.01105	0.00058
σ_m	0.00000	0.00000	0.00003	0.00000	0.00024	0.00001
σ_b	10.49812	0.55071	11.65861	0.61953	13.78634	0.72906

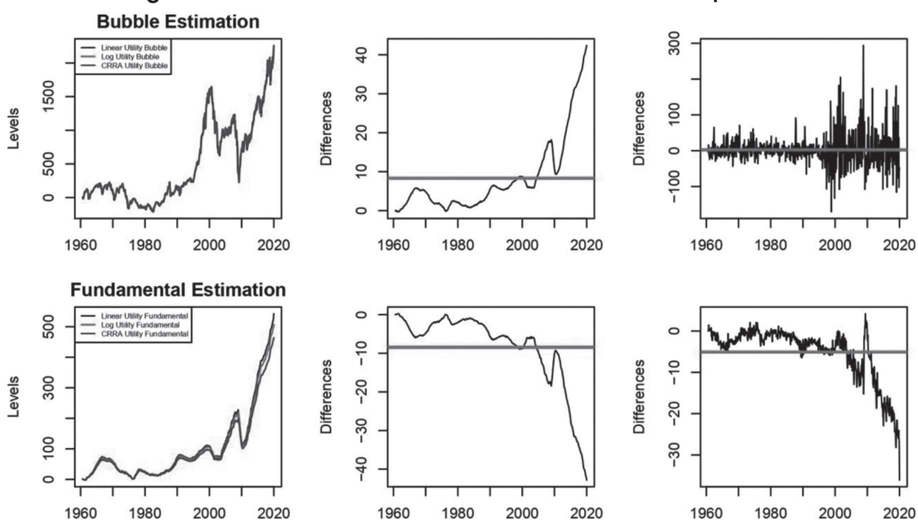
Source: Own elaboration.

We plot the three estimated bubbles $\tilde{b}_t$ and also the three estimated fundamentals $\tilde{f}_t$ and see how they behave for the different SDF m_t. As we mentioned earlier, we need an initial value for the estimated bubble $\tilde{b}_t$ and an initial value for the estimated fundamental $\tilde{f}_t$ since we only estimated $\Delta\tilde{b}_{t|t}$ and $\Delta\tilde{f}_t$. We consider conservative initial values of $\tilde{b}_1 = 1$ and $\tilde{f}_1 = 1$ and constructed the series. We are interested in the dynamics, not the levels per se.

We show the results in Figure 7. The three estimated bubbles are at the top left in the first plot and it looks like there are no differences among them. However, digging up closely, we consider the difference between the estimates of the CRRA Utility Bubble and the Log Utility Bubble. This is depicted in the middle graph of the top. We see that across all dates the CRRA utility bubble is bigger than the Log Utility Bubble. Finally, in the topright graph, we plot the difference between the Log Utility Bubble and the Linear Utility Bubble. At the beginning the difference is negible, but at later periods starting from 2000 the Log Utility Bubble turns bigger than the Linear Utility Bubble.

We plot the three estimated fundamentals at the bottomleft graph and it is more clear that the CRRA Utility Fundamental is the lowest, especially from 2010 onwards, being in the middle the Log Utility Fundamental and finally the at the top the Linear Utility Fundamental. This is confirmed in the last two graphs at the bottom. In the first one, we plot the difference between the CRRA Utility Fundamental and the Log Utility Fundamental which turns out to be negative for the whole sample period. The same is true for the difference between the Log Utility Fundamental and the Linear Utility Fundamental plotted in the bottomright panel.

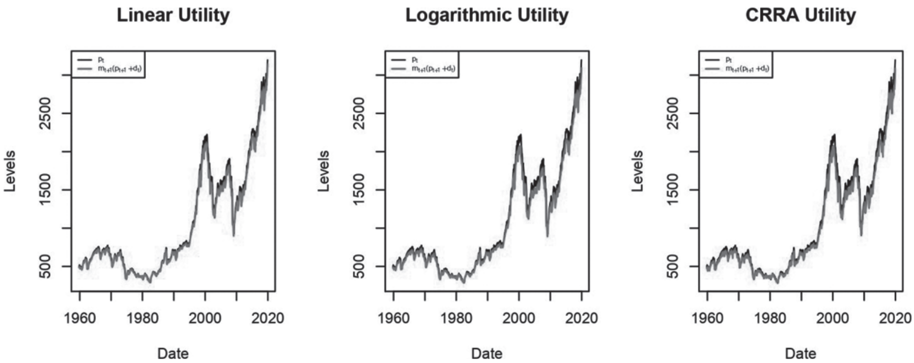
Figure 7: Estimated Bubbles and Fundamentals: a comparison



Source: Own elaboration.

Thus we have the result that the bigger the fundamental value the lower the bubble. In our model two variables drive fundamental value: dividends d_t and the SDF m_t. According to our results shown in Table 6, it looks like empirically, the dynamics of the fundamental are explained by dividends, and the size of the fundamental is determined by the interaction between the dividends and the SDF. This is through the median of the SDF we considered in the analysis $\overline{m}$. For the Linear Utility Model $\overline{m}_{LU} = 0.93$, for the Log Utility Model $\overline{m}_{Log} = 0.92$ and finally for the CRRA utility model $\overline{m}_{CRRA} = 0.90$, thus the lower the SDF the lower the fundamental value and the higher the bubble. In fact, we are not the first to arrive this conclusion. Galí (2014) shows rigorously that a *leaning against the wind* Monetary Policy is not favorable if one looks to reduce a bubble. Let us consider this in a model with linear utility and no borrowing constraint. Thus, $m_t = \frac{1}{1+r_t}$. If the Central Bank increases r_t then m_t decreases and the fundamental value decreases leaving room for a bigger bubble. This is confirmed by the results obtained in this analysis and it is at odds with conventional wisdom that increasing interest rates helps reduce a bubble. Given the important role of the SDF in the size of the bubble estimated for the S&P500, we analyze the validity of our main asset pricing equation 8 considering the three different types of SDFs analyzed. Figure 8 shows that the three different SDFs considered price reasonably well the S&P500.

Figure 8: Comparison of SDFs considered



Source: Own elaboration.

This is an important result because given the fact that we assumed constant parameters for impatience $\beta = 0.95$, financial openness $\phi = 0.5$, and risk aversion $\gamma = 3$; the resulting SDFs price reasonably well the S&P500.

Finally, it is worth digging up a bit further into the degree of financial openness. We gave a reasonable value for the computations of $\phi = 0.5$ but we can analyze what would happen if we are in an economy with a restrictive borrowing constraint, i.e., $\phi = 0.5$. For the Linear Utility $\overline{m}_{LU} = 0.95$, for the Log Utility $\overline{m}_{Log} = 0.94$ and for the CRRA Utility $\overline{m}_{CRRA} = 0.92$. Thus we would expect higher fundamental values in the three cases compared to the initial case of $\phi = 0.5$ and reduced bubble sizes[10]. Therefore, it would seem that putting restrictions to credit can help reduce a bubble. At least in the case of the S&P500.

4. Conclusions

We went deep into questioning the relationship between Rational Bubbles and the S&P500. The evidence suggests that the dynamics of the S&P500 resembles one of a rational bubble and this conclusion is robust to the different SDFs considered. This is important for institutional investors and pension funds who invest in the stock market since they can use the rational bubble equation to forecast the expected returns when considering strategic allocations. For Central Banks, it gives the message that the stock market is driven by animal spirits and fundamental components are not that crucial in this market. The fundamental component of the S&P500 is mainly driven by dividends d_t and the median of the SDF $\overline{m}$. Particularly, the dividends series give the dinamycs of the fundamental value but ultimately $\overline{m}$ defines the size. Thus a lower SDF results in a bigger bubble. Among our estimates, the CRRA utility SDF gives the biggest bubble. Finally, we briefly analyzed the impact of the borrowing restrictions on the size of a bubble in the S&P500 case and we concluded that being restrictive in credit reduces the size of a bubble. This is another important result for Central Banks and Financial Authorities, they can restrict borrowing from stock market investors if a considerable bubble size is perceived.

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10 We do not do this computation since we think the relationship between the SDF and bubble size has been already exposed.

References

1. Diba, B. T. and Grossman, H. I. (1988). The theory of rational bubbles in stock prices. *The Economic Journal, 98*(392), 746-754.
2. Evans, G. W. (1991). Pitfalls in testing for explosive bubbles in asset prices. *The American Economic Review, 81*(4), 922-930.
3. Galí, J. (2014). Monetary policy and rational asset price bubbles. *American Economic Review, 104*(3), 721-752.
4. Gürkaynak, R. S. (2008). Econometric tests of asset price bubbles: Takingstock*. *Journal of Economic Surveys, 22*(1), 166-186.
5. Martin, A. and Ventura, J. (2016). Managing credit bubbles. *Journal of the European Economic Association, 14*(3), 753-789.
6. Norden, S. V. (1996). Regime switching as a test for exchange rate bubbles. *Journal of Applied Econometrics, 11*(3), 219-251.
7. Phillips, P. C. B., Wu, Y. and Yu, J. (2011). *Explosive behavior in the1990s nasdaq: When did exuberance escalate asset values?* International Economic Review, 52*(1), 201-226.
8. Santos, M. S. and Woodford, M. (1997). Rational asset pricing bubbles. *Econometrica, 65*(1), 19-57.
9. Sims, C. A. (1988). Bayesian skepticism on unit root econometrics. *Journal of Economic Dynamics and Control, 12*(2), 463-474.
10. So, B. S. and Shin, D. W. (2001). An invariant sign test for random walks based on recursive median adjustment. *Journal of Econometrics, 102*(2), 197-229.
11. Uribe, J. M. and Mejía, J. F. (2014). Burbujas financieras y comportamiento reciente de los mercados de acciones en América Latina. *Lecturas de Economía,* 81,57-90.
12. Wu, Y. (1997). Rational bubbles in the stock market: Accounting for the u.s. stock - price volatility. *Economic Inquiry, 35*(2), 309-319.

Annex

Derivation of Linear Utility Asset Pricing Equation

In this appendix we will derive the equation 14. To begin with, we found out that dividends d_t follow a process ARIMA$(4,1,0)$ *i.e.*

$$\Delta d_t = \alpha_1 \Delta d_{t-1} + \alpha_2 \Delta d_{t-2} + \alpha_3 \Delta d_{t-3} + \alpha_4 \Delta d_{t-4} + u_t \tag{21}$$

We can rewrite 21 as an AR(1) process:

$$\underbrace{\begin{bmatrix} \Delta d_t \\ \Delta d_{t-1} \\ \Delta d_{t-2} \\ \Delta d_{t-3} \end{bmatrix}}_{Y_t} = \underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \Delta d_{t-1} \\ \Delta d_{t-2} \\ \Delta d_{t-3} \\ \Delta d_{t-4} \end{bmatrix}}_{Y_{t-1}} + \underbrace{\begin{bmatrix} u_t \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{U_t} \tag{22}$$

Updating 22 one period we have:

$$Y_{t+1} = AY_t + U_{t+1} \tag{23}$$

Forward updating equation 23 until period $t+j$:

$$Y_{t+j} = A^j Y_t + \sum_{i=0}^{j-1} A^i U_{t+j-i} \tag{24}$$

Multiplying 24 by the vector (1x4) $J = [1,0,0,0]$ and applying the $E_t[\ ]$ operator, we have:

$$\begin{aligned}
E_t\left[d_{t+j}\right] &= E_t\left[d_{t+j-1}\right] + JA^jY_t \\
E_t\left[d_{t+j}\right] &= E_t\left[d_{t+j-2}\right] + JA^{j-1}Y_t + JA^jY_t \\
&\vdots \\
E_t\left[d_{t+j}\right] &= d_t + J\left(\sum_{i=1}^{j} A^i\right)Y_t
\end{aligned} \tag{25}$$

Thus:

$$E_t\left[d_{t+j}\right] = d_t + \left(I - A\right)^{-1} A\left(I - A^j\right)Y_t \tag{26}$$

In the same way, we can start at $Y_t = AY_{t-1} + U_t$ to arrive to:

$$E_{t-1}\left[d_{t+j-1}\right] = d_{t-1} + \left(I - A\right)^{-1} A\left(I - A^j\right)Y_{t-1} \tag{27}$$

Therefore:

$$E_t\left[d_{t+j}\right] - E_{t-1}\left[d_{t+j-1}\right] = \Delta d_t + \left(I - A\right)^{-1} A\left(I - A^j\right)\Delta Y_t \tag{28}$$

Similarly, an ARIMA$(3,0,0)$ process was found for m_t. We can rewrite it as an AR(1) process:

$$\underbrace{\begin{bmatrix} m_t \\ m_{t-1} \\ m_{t-2} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}}_{C} + \underbrace{\begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} m_{t-1} \\ m_{t-2} \\ m_{t-3} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} v_t \\ 0 \\ 0 \end{bmatrix}}_{V_t} \tag{29}$$

Starting with equation 29 and repeating the process we did for d_t considering a vector (1x3) $K = \left[1,0,0\right]$, we obtain:

$$E_t\left[m_{t+j+1}\right]-E_{t-1}\left[m_{t+j}\right]=KBB^j\Delta X_t \qquad (30)$$

Substituing 29and 30 into the price equation 12, we have:

$$\Delta p_t=\frac{\bar{m}}{1-\bar{m}}\Delta d_t+\bar{m}J\left(I-A\right)^{-1}A\left(\frac{1}{1-\bar{m}}I-\left(I-\bar{m}A\right)^{-1}\right)\Delta Y_t+\left(\bar{p}+\bar{d}\right)KB\left(I-\bar{m}B\right)^{-1}\Delta X_t+\Delta b_t \qquad (31)$$

It is a matter of algebra to arrive from equation 30 to equation 14.