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Special Sheffer-operators on a p-valued logic

Operadores de Sheffer especiales en una lógica p-valuada

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Abstract: We studied the Sheffer-operators that can be written under the form $1+\alpha(r,s)$ where α is an associative, commutative and idempotent operator on $Z_p[x,y]$, with p prime. We conjecture that such operators are always Sheffer operators. We show the conjecture is true for $p=2$ and $p=3$.

Key words: Logic, Multivalued, Sheffer.

Resumen: Se estudia los operadores de Sheffer que pueden escribirse bajo la forma $1+\alpha(r,s)$, donde α es una operación asociativa, conmutativa e idempotente sobre $Z_p[x,y]$, con p primo. Conjeturamos que tales operadores son siempre operadores de Sheffer y mostramos que la conjetura es cierta para $p=2$ y $p=3$.

Palabras Clave: Lógica, Multivaluada, Sheffer.

1 Introduction

In a previous article [1] we have shown that every operator in a p-valued logic can be written as a polynomial in $Z_p[x,y]$. We have also remarked that three of the most known multivalued Sheffer-operators can be written as $1+\alpha(x,y)$, where α is associative, commutative and idempotent on its variables. This is the case of Post-operator $inc(\min(x,y))$, the Webb-operator $inc(\max(x,y))$ and the Webb stroke function $|$, whose polynomial form is δ[1].

For instance, if $p=3$, the three mentioned operators are:

[1] δ is a Sheffer-operator. Donald L. Webb showed it in 1935 [3]. We rewrote the proof under an algebraic point of view [2].

$$\begin{bmatrix}1&1&1\\1&2&2\\1&2&0\end{bmatrix}=1+\begin{bmatrix}0&0&0\\0&1&1\\0&1&2\end{bmatrix}\ ,\ \begin{bmatrix}1&2&0\\2&2&0\\0&0&0\end{bmatrix}=1+\begin{bmatrix}0&1&2\\1&1&2\\2&2&2\end{bmatrix}\ ,$$

$$\begin{bmatrix}1&1&1\\1&2&1\\1&1&0\end{bmatrix}=1+\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}$$

whose polynomial expressions[2] are:

$$\mu(x,y)=1+\alpha_{\wedge}(x,y)=1+xy+2x^2y+2xy^2+2x^2y^2$$

$$\omega(x,y)=1+\alpha_{\vee}(x,y)=1+x+y+2xy+x^2y+xy^2+x^2y^2$$

$$\delta(x,y)=1+\alpha_{|}(x,y)=1+2x^2y+2xy^2$$

Of course, these three operators are not the only ones that have the properties indicated above. How many there are? Which are they? Are all the operators with these properties Sheffer-operators?

2 A theorem on special Sheffer-operators

To answer these questions, we introduce an action of the S_p group on the set $Z_p[x,y]$ defined by

$$(\theta*\gamma)(x,y)\coloneqq\theta^{-1}(\gamma(\theta(x),\theta(y))$$

Where $\theta\in S_p$. It is clear that $\theta*\gamma$ is in $Z_p[x,y]$ if γ is in $Z_p[x,y]$.[3]

Let's take $S_p=<\rho;\tau>$, where ρ is the first *p*-cycle and τ is the transposition between 0 and 1. For instance, if $p=3$, we have $\rho(x)=1+x$ and $\tau(x)=1+2x$.

Theorem 1 Let be $\theta\in S_p$ and $\gamma\in Z_p[x,y]$.

$\theta*\gamma$ is a Sheffer operator if γ it is.

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[2] The way we build the polynomial expression of a logic operator is explained in detail in [2].

[3] You can easily see that * thus defined is an action *à droite* of a group on a set.

Proof

For an operator $\gamma(x, y) \in Z_p[x, y]$, define $Im(\gamma)$ in a recursive way:

1) $x \in Im(\gamma)$
2) $y \in Im(\gamma)$
3) $r, s \in Im(\gamma) \Longrightarrow \gamma(r, s) \in Im(\gamma)$

Then γ is a Sheffer-operator if and only if $Im(\gamma) = Z_p[x, y]$.

Now, if γ is a Sheffer-operator[4], it is easy to see that $x, y \in Im(\theta * \gamma)$. Indeed $\theta * x = x$ and $\theta * y = y$.

Now let be r and s in $Im(\gamma)$. Then $\gamma(r, s) \in Im(\gamma)$. Since γ is a Sheffer operator we have that $\theta^{-1}(r)$ and $\theta^{-1}(s)$ are in $Im(\gamma)$. So we can use the identity:

$$\begin{aligned}\gamma(\theta^{-1}(r), \theta^{-1}(s)) &= \theta^{-1} * \left((\theta * \gamma)(\theta^{-1}(r), \theta^{-1}(s))\right) \\ &= \theta^{-1}\left((\theta * \gamma)(\theta(\theta^{-1}(r)), \theta(\theta^{-1}(s)))\right)\end{aligned}$$

$$\Longrightarrow \quad \theta\left(\gamma(\theta^{-1}(r), \theta^{-1}(s))\right) = (\theta * \gamma)(r, s)$$

That is if $\gamma(r, s) \in Im(\gamma)$ then $\gamma(\theta^{-1}(r), \theta^{-1}(s)) \in Im(\gamma)$ and then:

$$\theta\left(\gamma(r, s)\right) \in Im(\theta * \gamma)$$

Like $\theta: Z_p[x, y] \rightarrow Z_p[x, y]$ is a one to one map, whose inverse is θ^{-1}, we have

$$Im(\theta * \gamma) = Z_p[x, y]$$

Theorem 2

If $\alpha \epsilon Z_p[x, y]$ is associative, so $\theta * \alpha$ it is.

If $\alpha \epsilon Z_p[x, y]$ is commutative, so $\theta * \alpha$ it is.

If $\alpha \epsilon Z_p[x, y]$ is idempotent, so $\theta * \alpha$ it is.

[4] Remember that a logic operator γ is called a Sheffer operator if and only if all the logic operators may be written using only γ.

Proof

1. Suppose that $\alpha \in Z_p[x,y]$ is associative. It means $\alpha\big(r,\alpha(s,t)\big) = \alpha(\alpha(r,s),t)$ for all $r,s,t \in Z_p[x,y]$.

Then

$$(\theta * \alpha)\big(r,(\theta * \alpha)(s,t)\big) = \theta^{-1}\Big(\alpha\Big(\theta(r),\theta\big((\theta * \alpha)(s,t)\big)\Big)\Big)$$

$$= \theta^{-1}\left(\alpha\left(\theta(r),\theta\left(\theta^{-1}\Big(\alpha\big(\theta(s),\theta(t)\big)\Big)\right)\right)\right)$$

$$= \theta^{-1}\Big(\alpha\Big(\theta(r),\alpha\big(\theta(s),\theta(t)\big)\Big)\Big)$$

$$= \theta^{-1}\Big(\alpha\Big(\alpha\big(\theta(r),\theta(s)\big),\theta(t)\Big)\Big)$$

$$= \theta^{-1}\left(\alpha\left(\theta\left(\theta^{-1}\Big(\alpha\big(\theta(r),\theta(s)\big)\Big)\right),\theta(t)\right)\right)$$

$$= \theta^{-1}\Big(\alpha\Big(\theta\big((\theta * \alpha)(r,s)\big),\theta(t)\Big)\Big)$$

$$= (\theta * \alpha)\big((\theta * \alpha)(r,s),t\big)$$

2. Suppose that $\alpha \in Z_p[x,y]$ is commutative. It means $\alpha(r,s) = \alpha(s,r)$ for all $r,s \in Z_p[x,y]$.

$$(\theta * \alpha)(r,s) = \theta^{-1}\big(\alpha(\theta(r),\theta(s))\big) = \theta^{-1}\big(\alpha(\theta(s),\theta(r))\big) = (\theta * \alpha)(s,r)$$

3. Suppose that $\alpha \in Z_p[x,y]$ is idempotent. It means $\alpha(r,r) = r$ for all $r \in Z_p[x,y]$.

$$(\theta * \alpha)(r,r) = \theta^{-1}\big(\alpha(\theta(r),\theta(r))\big) = \theta^{-1}(\theta(r)) = r$$

Corollary

Let be
$A_p = \{\, \alpha \epsilon Z_p[x,y] \mid \alpha \textit{ is asociative, comutative and idempotent}\}$.

If $\alpha \in A_p$ then $\theta * \alpha \in A_p$.

Cases $\boldsymbol{p = 3}$ and $\boldsymbol{p = 5}$

For $p = 3$ we have nine polynomials of the form $1 + \alpha(x, y)$ with $\alpha \in A_3$. α must be one of:

$$\alpha_1(x, y) = 2x^2y + 2xy^2$$

$$\alpha_2(x, y) = xy + 2x^2y + 2xy^2 + 2x^2y^2$$

$$\alpha_3(x, y) = 2xy + 2x^2y + 2xy^2 + x^2y^2$$

$$\alpha_4(x, y) = 2x + x^2 + 2y + 2xy + y^2 + 2x^2y^2$$

$$\alpha_5(x, y) = 2x + 2x^2 + 2y + xy + 2y^2 + x^2y^2$$

$$\alpha_6(x, y) = x^2 + xy + 2x^2y + y^2 + 2xy^2$$

$$\alpha_7(x, y) = x + y + xy + 2x^2y + 2xy^2 + 2x^2y^2$$

$$\alpha_8(x, y) = x + y + 2xy + x^2y + xy^2 + x^2y^2$$

$$\alpha_9(x, y) = 2x^2 + 2xy + 2x^2y + 2y^2 + 2xy^2$$

We see that $\alpha_1 = \alpha_|$, $\alpha_2 = \alpha_\wedge$ and $\alpha_7 = \alpha_\vee$. All nine $1 + \alpha_i(x, y)$ are Sheffer operators. We have two orbits:

$$\{\alpha_1;\ \alpha_6;\ \alpha_9\}\ and \{\alpha_2;\ \alpha_3;\ \alpha_4;\ \alpha_5;\ \alpha_7;\ \alpha_8\}.$$

Using a computer, for $p = 5$ we have found 1065 Sheffer operators of this special form. We conjecture that if an operator has this special form then it is a Sheffer operator. This assumption is true for little values of p prime. For instance, for $p = 2$ we have only two Sheffer operators $1 + xy$ and $1 + x + y + xy$, the Sheffer stroke and the Pierce arrow and, in this case, it is very easy to show that the conjecture is true. For $p = 3$ we are in a similar situation, because under the conditions imposed to $\alpha \epsilon A_3$ we found that $1 + \alpha(x, y)$ is one of the nine Sheffer operators listed above. Indeed…

Theorem 3

If $\gamma(x, y) = 1 + \alpha(x, y)$ is an operator of a 3-valued logic with $\alpha \epsilon A_3$, γ is a Sheffer operator.

Proof

Since α is commutative and idempotent, the matrix form[5] of α is $\begin{bmatrix} 0 & a & b \\ a & 1 & c \\ b & c & 2 \end{bmatrix}$ with $a, b, c \in Z_p$. Of course, we have $\alpha(0,1) = a$, $\alpha(0,2) = b$ and $\alpha(1,2) = c$. There are 27 cases.

If $a = 2$ then $b = 2$ and $c = 2$, because

$$2 = a = \alpha(0,1) = \alpha(\alpha(0,0), 1) = \alpha\big(0, \alpha(0,1)\big) = \alpha(0,2) = b$$

$$2 = a = \alpha(0,1) = \alpha\big(0, \alpha(1,1)\big) = \alpha(\alpha(0,1), 1) = \alpha(2,1) = \alpha(1,2) = c$$

By the same way we can show that:

if $b = 1$ then $a = 1$ and $c = 1$

if $c = 0$ then $a = 0$ and $b = 0$

So we have the following three operators in A_3

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} ; \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} ; \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

On the other hand we can exclude all the operators failing the rules showed above. There are $1 \cdot 3 \cdot 3 + 2 \cdot 1 \cdot 3 + 2 \cdot 2 \cdot 1 - 3 = 16$ of them. They all are not associative.

There are eight operators left to study. For them we have $a \in \{0; 1\}$, $b \in \{0; 2\}$, $c \in \{1; 2\}$.

We see there are two cases that are clearly not associative: $a = 0,\ b = 2,\ c = 1$ and $a = 1,\ b = 0,\ c = 2$ because

$$\alpha(\alpha(0,1), 2) = \alpha(0,2) = 2 \text{ but } \alpha\big(0, \alpha(1,2)\big) = \alpha(0,1) = 0$$

and

$$\alpha(\alpha(1,0), 2) = \alpha(\alpha(0,1), 2) = \alpha(1,2) = 2 \text{ but } \alpha\big(1, \alpha(0,2)\big) = \alpha(1,0) = \alpha(0,1) = 1$$

[5] See [2].

The six operators remaining are:

$$\begin{bmatrix}0&0&0\\0&1&1\\0&1&2\end{bmatrix};\begin{bmatrix}0&0&0\\0&1&2\\0&2&2\end{bmatrix};\begin{bmatrix}0&0&2\\0&1&2\\2&2&2\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\1&1&1\\0&1&2\end{bmatrix};\begin{bmatrix}0&1&2\\1&1&1\\2&1&2\end{bmatrix};\begin{bmatrix}0&1&2\\1&1&2\\2&2&2\end{bmatrix}$$

They are all six associative.

We recognize the two orbits formed by the action of S_3 on A_3. In the first orbit there is the well-known operator $\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}$ corresponding to $\alpha_{|}(x,y)$. Since $1+\alpha_{|}(x,y)$ is the Webb stroke and it is a Sheffer operator, all the operators in its orbit define also Sheffer operators.

In the second orbit we have another operator we know quite well: $\begin{bmatrix}0&0&0\\0&1&1\\0&1&2\end{bmatrix}$. It corresponds to $\alpha_{\wedge}(x,y)$. Like $1+\alpha_{\wedge}(x,y)$ is the Post-operator and it is a Sheffer operator we have again that all the operators in its orbit define Sheffer operators.

Bibliography

[1] Pino O., Morales Z. (2015) *Un operador de Sheffer en la Lógica IGR_3.* Acta Nova, Vol 7, Nº1. Cochabamba, Bolivia.

[2] Pino O. (2018) *Un operador de Sheffer en la Lógica IGR_p.* Acta Nova, Vol 8, Nº4. Cochabamba, Bolivia.

[3] Webb D. L. (1935). *Generation of any n-valued logic by one binary operation*. Proceedings National Academy of Sciences. U.S.A. May 1935.

[4] Stojmenovic I., (1988). *On Sheffer symmetric functions in three valued logic*. Discrete Applied Mathematics 22, North-Holland.

[5] Foxley E., (1962) *The determination of all Sheffer functions in 3-valued logic, using a logical computer*. Notre Dame Journal of Formal Logic, Volume III, Number 1. Nottingham, England.